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A Nonlinear Theory of Wake Development

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SUMMARY
A simple new series, using an expansion of the velocity profile in parabolic cylinder functions, has been developed to describe the nonlinear evolution of a steady, laminar, incompressible wake from a given arbitrary initial profile. The first term in this series is itself found to provide a very satisfactory prediction of the decay of the maximum velocity defect in the wake behind a flat plate or aft of the recirculation zone behind a symmetric blunt body. A detailed analysis, including higher order terms, has been made of the flat plate wake with a Blasius profile at the trailing edge. The same method yields, as a special case, complete results for the development of linearized wakes with arbitrary initial profile under the influence of arbitrary pressure gradients. Finally, for purposes of comparison, a simple approximate solution is obtained using momentum integral methods, and found to predict satisfactorily the decay of the maximum velocity defect.

1. Introduction

There has been increasing interest in recent years in studies of the development and dynamics of wakes behind bodies. In the far wake, i.e. at large distances downstream where the velocity defects in the wake are small, the flow is governed by linear equations and has been studied in considerable detail, beginning with the early investigations of Tollmien (see Schlichting [1]) and Goldstein [2]. In the near wake, however, the flow is dominated by separation and reattachment or trailing edge singularities, and is not yet completely understood. In between these extremities is a large region of nonlinear wake flow, which is the subject of the present work. Available evidence indicates that the near wake, especially in the case of a flat plate with a sharp trailing edge (which will be our major concern in the following), exerts only a local influence, and possibly only sets certain initial conditions which then determine the wake development independently. It is therefore of interest to be able to calculate the development of the wake from a given (arbitrary) initial profile.

The first attempts to obtain a complete description of the wake behind a flat plate were also made in the classic work of Goldstein [2, 3] who constructed asymptotic expansions of the solution respectively at small and at large values of the downstream coordinate x (in the form of power series in x). However, he found that beyond the second term the asymptotic far wake solution at large x developed divergences (see Stewartson [4] on this point). Although the two expansions valid for $x \to 0$ and $x \to \infty$ still left a large gap in the middle, a suitable translation (in x) of the far wake solution enabled Goldstein to patch it with the near wake solution at some intermediate point.

Various experimental observations indicate that the shape of the velocity distribution behind a symmetric body settles down to a Gaussian remarkably rapidly. One way to exploit this feature in a theoretical attack on the problem is to assume a Gaussian velocity profile with certain free parameters which are then evaluated by a momentum integral technique. Calculations using such a method (described in Appendix 2 and applied to a flat plate wake) do often give simple and reasonable results, as in the present case. However, the errors in an integral method cannot always be satisfactorily estimated, and furthermore cannot be reduced by any systematic procedure. We propose here (in section 3) a new method which is both simple and accurate, and can in principle be used in a wide variety of shear flow problems of the kind discussed by Charwat and Der [5]. In this method, the velocity distribution at each station is expanded in a series of orthogonal functions, and equations of motion are formulated for the Fourier coefficients in such an expansion, i.e., for the spectrum of the velocity distribution. This leads to an infinite system of ordinary differential equations for the Fourier coefficients; these equations are coupled and nonlinear in general. But the fact that one mode dominates most of the flow makes it possible to solve these equations relatively easily, and all the Fourier amplitudes can be calculated to great accuracy; indeed, we have reason to believe that our final results are more accurate than the numerical solution based on finite difference schemes [5, 6]; for this reason, the present approach can be looked upon both as a simple approximate method (when very few terms are used in the series) and as an efficient numerical method capable of great accuracy when more terms are included in the series.

Because we use an orthogonal expansion for the velocity profile, the present approach may appropriately be called a spectral evolution or nonlinear mode interaction theory. At large distances downstream the problem becomes linear, as already pointed out; the modes then decouple to some extent, and we easily obtain a simple description of the far wake (section 5). A general analysis of the linear far wake has already been given by Gold [11], but the present work provides an alternative approach which is instructive.

2. Formulation of the Problem

Figure 1 gives a schematic representation of the problem under consideration. The velocity profile at the given initial station $x=0$ in the flow field is prescribed say as being $\bar{u}_0(\bar{y})$, and it is desired to find the downstream evolution of the profile. In the present paper only symmetric profiles are considered but the extension to nonsymmetric profiles is natural to the method we employ. We assume that the development of any profile, which will henceforth be termed a wake for convenience (although jet and other shear flows could also be treated by the same

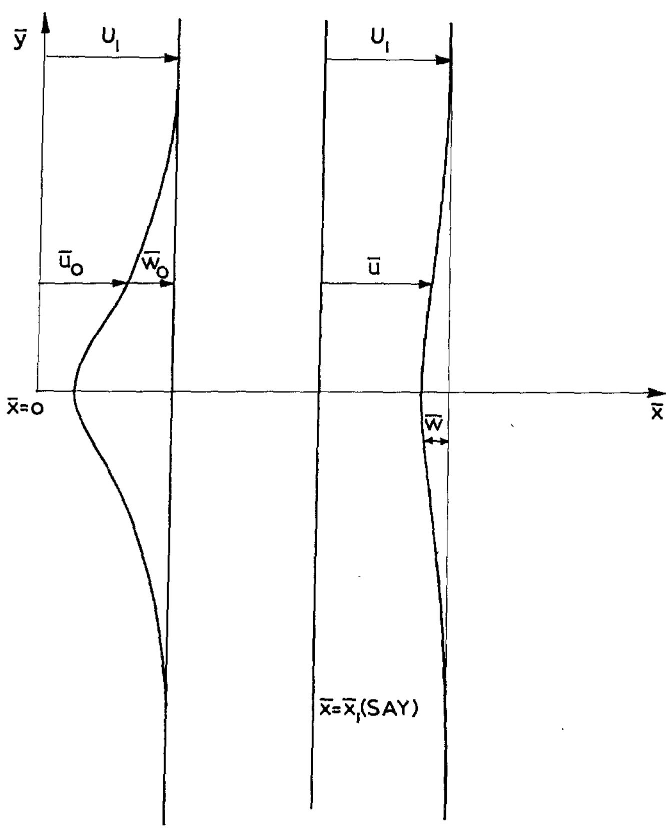

Figure 1. Schematic diagram of the problem.

method), is governed by the boundary-layer equations. For laminar, incompressible flow in the absence of pressure gradient they can be written as

$$(\bar{u}\bar{y}^k)_{\bar{x}} + (\bar{v}\bar{y}^k)_{\bar{y}} = 0 \qquad \text{(continuity)} \tag{2.1a}$$

$$\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} = (\nu/\bar{y}^k)(\bar{u}_{\bar{y}}\bar{y}^k)_{\bar{y}} \qquad \text{(momentum)} \tag{2.1b}$$

where $\bar{x}$ and $\bar{y}$ are co-ordinates along and normal to the centre line of the wake, $\bar{u}$ and $\bar{v}$ are the velocities in the respective directions, ν the kinematic viscosity and k is an index which is equal to 0 for two-dimensional flow and 1 for axially symmetric flow. The subscripts indicate partial derivatives.

The initial conditions on the velocity profile, the boundary conditions on the centre line and the asymptotic condition at infinity in the normal direction are

$$\bar{u} = \bar{u}_0(\bar{y}) \text{ at } \bar{x} = 0$$

and

$$\left.\begin{array}{l} \bar{u}_{\bar{y}} = \bar{v} = 0 \text{ at } \bar{y} = 0 \\ \bar{u} \to U_1 \text{ as } \bar{y} \to \infty \end{array}\right\} \text{for all } \bar{x}\,. \tag{2.2}$$

On introducing the stream function $\bar{\psi}$ defined by

$$\bar{u}\bar{y}^k = \bar{\psi}_{\bar{y}}\,, \quad \bar{v}\bar{y}^k = -\bar{\psi}_{\bar{x}} \tag{2.3}$$

the continuity equation (2.1a) is identically satisfied. The momentum equation (2.1b) now reduces to

$$\bar{\psi}_{\bar{y}}\bar{\psi}_{\bar{x}\bar{y}} - \bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{y}\bar{y}} + (k/\bar{y})\bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{y}} = \nu\bar{y}^k(\bar{\psi}_{\bar{y}\bar{y}\bar{y}} - (k/\bar{y})\bar{\psi}_{\bar{y}\bar{y}} + (k/\bar{y}^2)\bar{\psi}_{\bar{y}})\,. \tag{2.4}$$

Equation (2.4) is now nondimensionalised by putting

$$\bar{\psi} = U_1\delta^{k+1}\psi(\eta, x)\,, \qquad \bar{u} = U_1 u,\ \bar{v} = U_1 v$$

$$\bar{y} = \eta\delta(x)/\sqrt{2}\,, \qquad \bar{x} = (U_1\delta_0^2/\nu)x \tag{2.5}$$

where ψ is the nondimensionalised stream function, x and η are the nondimensional distances in the axial and normal directions of the wake, U_1 the free stream velocity and δ is a length scale (still undefined). Subscript 0 indicates value at the initial station. With these transformations (2.4) reduces to

$$\psi_\eta\psi_{\eta x} - \psi_x\eta^k(\psi_\eta/\eta^k) - (k+1)(\delta'/\delta)\ \psi\eta^k(\psi_\eta/\eta^k)_\eta$$
$$= 2^{(1-k)/2}(\delta_0^2/\delta^2)\eta^k(\psi_{\eta\eta\eta} - k\psi_{\eta\eta}/\eta + k\psi_\eta/\eta^2)\,, \tag{2.6}$$

where $\delta' \equiv d\delta/dx$.

3. Method of Solution

3.1 *The Equations of Motion*

We now proceed to find the solution of (2.6) by assuming that the velocity defect profile at any station is given by a series of the form

$$w(x, \eta) \equiv 1 - u(x, \eta) = \sum_{p=0}^{\infty} F_p(x)D_{2p}(\eta) \tag{3.1}$$

where D_{2p} is the symmetric parabolic cylinder function of order $2p$ [7], defined by

$$D_{2p}(\eta) = e^{-\eta^2/4}(2p)!\,2^{-p}\sum_{m=0}^{p}\frac{(-1)^m(2\eta)^{2p-2m}}{m!\,(2p-2m)!} \tag{3.1a}$$

The functions D_{2p} satisfy the orthogonality condition

$$\int_0^\infty D_{2p}D_{2m}d\eta = 0 \quad \text{for} \quad p \neq m$$
$$= (2p)!\sqrt{\frac{\pi}{2}}\,, \quad p = m\,. \tag{3.1b}$$

It may be seen from (3.1) that the presence of only the D_0 component signifies that the profile is Gaussian.

Our choice of a series of parabolic cylinder functions to represent the velocity profile, instead of the oft tried solutions of the kind

$$u = 1 - \sum_n x^n \, G_n(\eta),$$

springs from a close inspection of the already known qualitative behaviour of the wake.

(a) All wake solutions should approach the free stream velocity exponentially as $\eta \to \infty$.

(b) The shear stress at the centre line of the wake should be zero, for all x. This implies that in laminar flow the velocity gradient in η direction should be zero at $\eta = 0$ for all x.

(c) The solution must asymptotically tend to the well known similarity solution as $x \to \infty$, regardless of the shape of the initial profile (as will be discussed in detail in § 5).

By the choice of (3.1) as the solution we are automatically satisfying these conditions on the complete solution. It is hoped that by such a choice the solution obtained for $F_p(x)$ and $\delta(x)$ will be valid over the entire range of x from 0 to ∞, which is a property not possessed by any previously known form of solution.

From (2.3) and (3.1), we can show that

$$2^{(k+1)/2} \psi_\eta = \eta^k (1 - F_p D_{2p}(\eta)) \; ; \tag{3.2}$$

here and in all subsequent equations it is convenient to adopt a summation convention by which a repeated index signifies summation over all integers from 0 to ∞. Introducing (3.2) in (2.6) and simplifying we get

$$\begin{aligned} & -F'_p D_{2p} + F_n F'_m ((B^p_{mn} - A^p_{mn}) D_{2p}) \\ & \qquad - \tfrac{1}{2}(\delta'/\delta) D_{2p} \{F_{p-1} + F_p + 2(p+1)(2p+1) F_{p+1} + 2(k+1) A^p_{mn} F_m F_n\} \\ & = -\tfrac{1}{2}(\delta_0^2/\delta^2)\{F_{p-1} - (4p+1)F_p + 2(p+1)(2p+1)F_{p+1} + 4k C^p_n F_n\} D_{2p}, \; F_{-1} = 0 \end{aligned} \tag{3.3}$$

where A^p_{mn}, B^p_{mn} and C^p_n are constants defined by

$$\eta^{-k} D'_{2n}(\eta) \int_0^\eta D_{2m}(\eta) \eta^k \, d\eta = A^p_{mn} D_{2p}(\eta) \tag{3.4a}$$

$$D_{2n}(\eta) D_{2m}(\eta) = B^p_{mn} D_{2p}(\eta) \tag{3.4b}$$

$$\eta^{-1} D'_{2n}(\eta) = C^p_n D_{2p}(\eta) \, . \tag{3.4c}$$

Dashes indicate differentiation with respect to appropriate variables. The constants A^p_{mn}, B^p_{mn} and C^p_n can be evaluated once and for all as they are independent of the initial profile. The procedure to evaluate these constants has been outlined in Appendix 1. Indeed, the one term solution involves only the following constants:

$$\begin{aligned} A^0_{00} &= -0.4082 \quad \text{for} \quad k = 0 \\ &= -0.182 \quad \text{for} \quad k = 1 \; ; \\ B^0_{00} &= \;\; 0.8165, \quad C^0_0 = -0.5 \quad (k = 0 \text{ or } 1) \end{aligned}$$

(3.3) is an equation involving $D_{2p}(\eta)$ and $F_p(x)$. The governing equation for $F_p(x)$ for any specified value of p, is obtained by multiplying (3.3) by $D_{2p}(\eta)$ and integrating in η between the limits 0 and ∞. Using the orthogonality relations (3.1a), we get the equation for $F_p(x)$ as

$$-F'_p + X^p_m F'_m - Y^p \delta' = -Z^p \tag{3.5}$$

where X^p_m, Y^p and Z^p are functions of the F_n and δ given by

$$X_m^p = F_n(B_{mn}^p - A_{mn}^p)$$
$$Y^p = (1/2\delta)\{F_{p-1} + F_p - 2(p+1)(2p+1)F_{p+1} + 2(k+1)A_{mn}^p F_m F_n\}$$
$$Z^p = \tfrac{1}{2}(\delta_0^2/\delta^2)\{F_{p-1} - (4p+1)F_p + 2(p+1)(2p+1)F_{p+1} + 4kC_n^p F_n\}\,. \tag{3.6}$$

The equations (3.5) form an infinite set of coupled nonlinear first order differential equations for the F_p with δ as an additional essentially arbitrary quantity. Because the equations are linear in the derivatives, they can be solved without too much difficulty by a marching process for any given set of initial values for the F_p.

3.2 *Initial and Boundary Conditions; Auxiliary Equation*

It can be observed from (3.1) that the boundary conditions on η have already been satisfied. We need only to satisfy the initial conditions on the F_p now. These can be obtained by expanding the initial profile $u_0(y)$ in a series of $D_{2p}(\eta)$ and using the orthogonality relations; we get

$$u_0(\eta) = 1 - F_p(0)D_{2p}(\eta) \tag{3.7}$$

$$F_p(0) = \int_0^\infty [1-u_0(\eta)]D_{2p}(\eta)d\eta \Big/ \int_0^\infty D_{2p}^2(\eta)d\eta\,. \tag{3.8}$$

To evaluate (3.8) one should however know the value of δ_0. To a certain extent, the choice of δ_0 (and indeed of δ, as we shall see shortly) is arbitrary and for any particular choice, the corresponding $F_p(0)$ can be evaluated using (3.8). This feature appears to be inherent in making a Fourier expansion of the type (3.7) over an infinite domain, as one is free to choose an arbitrary scaling parameter or unit in the independent variable. However, if we truncate (3.7) at some finite value of p as an approximation, δ_0 can in principle be taken as the value which in conjunction with the corresponding $F_p(0)$ gives the best approximation, say e.g. to hold the mean square error to a minimum.

We shall adopt here the following simpler alternative. First we determine the value of δ_0 such that a very small number of terms $F_p(0)$ represents the profile to a fair approximation, and subsequently include more terms in (3.7) (using the *same* value for δ_0) if higher accuracy is called for. E.g. in many practical applications as in the case of the wake behind a flat plate the profile is very nearly Gaussian and δ_0 can be conveniently chosen as the value of y where the defect velocity is equal to $e^{-\frac{1}{2}}$ times the maximum defect at the centre of the wake. Having chosen δ_0 we can now calculate $F_p(0)$ to represent the profile to any desired accuracy.

A similar problem arises at each value of x, and in fact we need one more relation prescribing the variation of δ with x in order to make the set of equations (3.5) complete. One possibility would be to take this from the momentum integral solution, but in this approach also (see Appendix 2) one needs an auxiliary equation in addition. We might therefore just as well use this auxiliary equation to complete the system (3.5). We have chosen this equation by requiring that the momentum equation be satisfied exactly along the centre-line by only the D_0 component of the velocity profile. This gives, from (3.5),

$$\frac{F_0 - 1}{F_0} F_0' = (k+1)\frac{\delta_0^2}{\delta^2}\,. \tag{3.9}$$

Other choices are possible, and it may be expected that any reasonable variation of δ with x should be satisfactory. However, some choices will be better than others; e.g. we have found that the adoption of the $\delta(x)$ distribution obtained from the linear solution leads to difficulties when the maximum velocity defect approaches unity.

4. Results in the Nonlinear Problem

4.1 *The one-term solution*

Before embarking on the computation of the wake development of an arbitrary profile, it is quite interesting and also instructive to study the development of a wake which has a Gaussian starting profile with large defect. It has been observed that most of the defect profiles, when allowed to develop undisturbed, settle down to an approximately exponential shape at a very short distance from the body producing the wake, even when the centre line defect itself may not be small in magnitude compared to the free stream velocity. However, the rate of decay of the centre line velocity defect is different from that exhibited by linear wakes, though the profiles are approximately the same. This can be observed by inspecting the calculations of Goldstein [3]. A similar trend is borne out by the experiments of Srinivasan [9] in turbulent flow. Hence we can expect that the analysis of the Gaussian profile gives a good first order approximation for the growth of wakes behind bodies after a very short distance from the trailing edge (or after a short distance downstream of the recirculation region in the case of blunt bodies). Furthermore such an analysis yields a simple solution for the development of the wake as we neglect higher order effects (i.e., all $F_n, n \neq 0$).

Retaining, then, only F_0 in (3.5) and (3.6), we get the equation for F_0 as

$$-F_0' + F_0 F_0' (B_{00}^0 - A_{00}^0) - (\delta'/\delta)\{(F_0/2) + (k+1) A_{00}^0 F_0^2\}$$
$$= \tfrac{1}{2}(\delta_0^2/\delta^2)(F_0 - 4kC_0^0 F_0) \tag{4.1}$$

With the use of the auxiliary equation (3.9) for δ, equation (4.1) can be simplified to give a relation between F_0 and δ as

$$F_0 \delta^{(1+k)} \{1 + 2(1+k) A_{00}^0 F_0\}^{\alpha_k} = \lambda_k \tag{4.2}$$

where

$$\alpha_k = (\tfrac{1}{2} + k/4 - B_{00}^0)\,. \tag{4.3}$$

λ_k is determined from the conditions at large x (F_0 small), when the left hand side of (4.3) can be evaluated from a consideration of the momentum integral.

The variation of F_0 with x can be obtained from (3.9) and (4.2) as

$$(\nu\bar{x}/U_1 \lambda_k^{2/1+k}) = \int_{F_0(0)}^{F_0} (F_0 - 1)\, dF_0 / [(k+1) F_0^{(k+3/(k+1)} \{1 + 2(1+k) A_{00}^0 F_0\}^{2\alpha_k/(1+k)}] \tag{4.4}$$

It can be easily seen that the solutions (4.2) and (4.4) reduce to the linear similarity solutions as $F_0 \to 0$,

$$F_0 \delta^{1+k} = \lambda_k \tag{4.5a}$$

$$F_0(x) = F_0(0)\{1 + (2\nu\bar{x}/U_1 \delta_0^2)\}^{-(1+k)/2}\,. \tag{4.5b}$$

Fig. 2 shows the "one term" solution (4.4) for the two-dimensional case along with many other results. Note that we have taken $F_0(0) = 1$, and that the abscissa is $\nu\bar{x}/U\theta^2$ where θ is the momentum thickness of the Blasius boundary layer at the trailing edge.

Fig. 3 shows a similar one-term solution, with $F_0(0) = 1$, for an axisymmetric wake.

Among the results from other workers shown in Fig. 2, those of Goldstein have already been mentioned. His "patched" solution is not shown to avoid cluttering the diagram, but it agrees well with the more accurate calculations of Charwat and Der [5], who solved the boundary layer equations numerically by a finite difference scheme. The accuracy of their calculations has been further improved [6], an indication of this being that the momentum thickness of the wake varies by less than 1% in their numerical solution (it should be strictly constant, by momentum conservation). We have also included in the diagram the experimental results of Hollingdale [8], which refer to the laminar wake behind a flat plate.

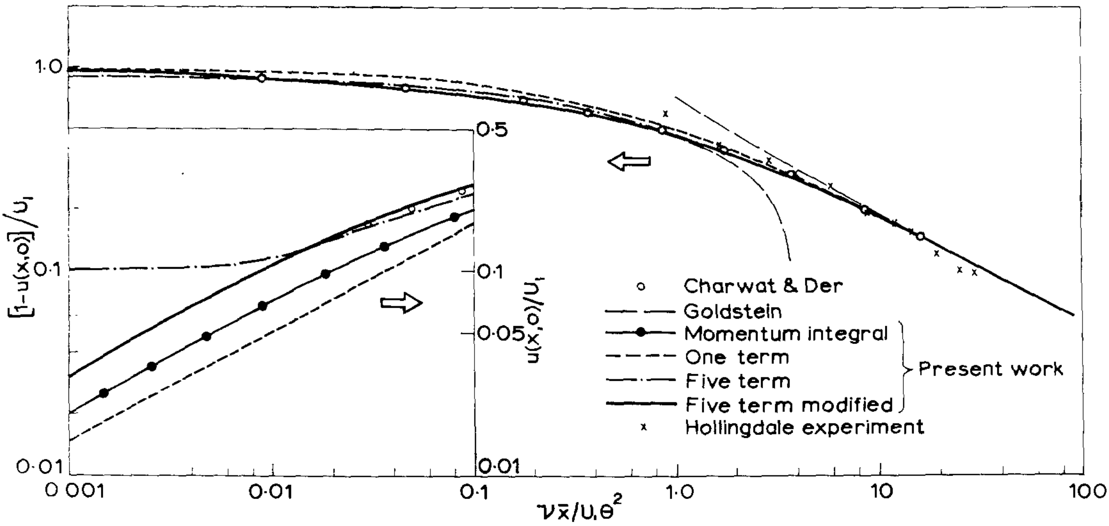

Figure 2. Decay of maximum defect velocity in two-dimensional flow, as predicted by various theories.

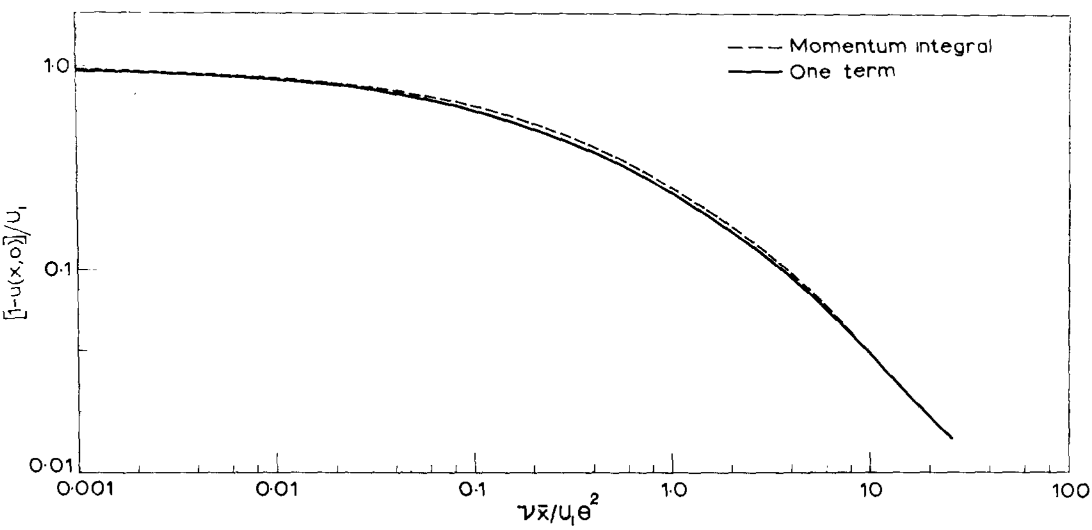

Figure 3. Decay of maximum defect velocity in axisymmetric flow.

It is surprising how well the simple solution (4.4) predicts the decay of the maximum velocity defect along x. However at very small values of x the agreement is not very good (see inset to Fig. 2), perhaps due to the strong assumption made on the initial velocity profile. Thus, an analysis of the Blasius profile at $x=0$, as in (3.7), shows that nearly 30 % of the total momentum defect is contained in the higher components F_1, F_2 ..., which have been ignored in the simple solution. Furthermore we have used the exact Blasius value of θ in comparing different solutions, although the assumption $F_0(0)=1$ is not strictly consistent with it. We now proceed to make a more refined analysis taking account of these higher order terms.

4.2. *Higher Order Effects*

The Blasius profile has been taken as the initial profile for the study of higher order effects as detailed results are available for comparison only in this case (Goldstein, Charwat *et al.*). It is found from an analysis made using (3.8) that the appropriate initial values for F_p are:

$$F_0(0) = 0.9822 \qquad F_1(0) = 0.1176$$
$$F_2(0) = 0.0107 \qquad F_3(0) = 0.00007 \tag{4.6}$$
$$F_4(0) = 0.00001 \qquad \delta_0 = 0.6\,.$$

The higher components are negligibly small; we therefore truncate the series (3.1) at $p=4$, and solve the equations (3.5) for F_0 to F_4 with (4.6) as the initial conditions and (3.9) as the auxiliary equation. This constitutes a marching problem, and was easily solved on an Elliott 803 computer using a standard Runge–Kutta routine. The corresponding solution is also shown in Fig. 2, where it is labelled the "five-term solution". It may be observed from the figure that the maximum defect in the initial profile reaches an asymptotic value of 0.9 only, as x tends to 0. This is due to the fact that by the choice of the $F_p(0)$ as in (4.6) the centre line defect is not accurately given at the starting point. However, the effect of this is only local; for $x > 0.001$ the predicted wake development shows excellent agreement with the curves given by Charwat and Der [5]. For the sake of clarity only a few representative points from the computations of [5] are shown in Fig. 2, as the differences are too small to be noticeable on a graph. It is found that in our solutions the momentum thickness is constant to within 0.1% compared to 1% in [3] as can be judged by the following table.

$\nu\bar{x}/U_1\theta^2$	0	0.33	32.76
$1-u/U_1$	0.9	0.6388	0.1014
θ	0.4390	0.4395	0.4392

So as to study the effect of keeping the maximum initial defect also at the correct value of unity without altering the momentum defect the following compromise was made. We keep the $F_0(0)$, $F_1(0)$ and $F_2(0)$ values as obtained by (3.6), but calculate the values of $F_3(0)$ and $F_4(0)$ such that both the centre line defect and momentum defect are correctly represented. We then find that the values of $F_3(0)$ and $F_4(0)$ have to be changed from those given in (4.6) to

$$F_3(0) = -0.0014\,, \quad F_4(0) = 0.00078\,. \tag{4.7}$$

With (4.7) as the initial profile the development of the wake downstream was calculated as before and the result is shown in Fig. 2 as the "five term modified" solution. It is obvious that this solution gives an excellent prediction of wake development all the way from $x=0$ to $x\to\infty$.

5. Linearised Wakes

We have seen in § 3 and 4 how the development of an arbitrary profile with large defect can be analysed. It can be shown that if the defect is small, so that the equations can be linearised, the solution for the development of any arbitrary shape can be expressed in closed form. For the sake of brevity we describe here the two-dimensional case only. However the solution for the axisymmetric case can be worked out on identical lines. The differences are pointed out at the end of the present section.

Without much difficulty one can even include the effect of arbitrary pressure gradients in analyzing linearized wakes; the relevant equation of motion, for a two-dimensional laminar wake with a small defect, is [10]:

$$(\delta_0^2/\delta^2)w_{\eta\eta} + \eta w_\eta (U\delta)_x/\delta - (U\delta)_x = 0 \tag{5.1}$$

with boundary conditions

$$w_\eta = 0 \text{ at } \eta = 0; \quad w \to 0 \text{ as } \eta \to \infty$$
$$w = 1 - u_0(\eta) \text{ at } x = 0\,.$$

It may be noted that all velocities are now non-dimensionalized with respect to some reference velocity U_1, and $U(x)$ is the free stream velocity at x non-dimensionalised with respect to the same reference velocity.

We now expand w as a series in parabolic cylinder functions as in (3.1). Carrying out simplifications as in § 3 we obtain the equations governing the different F_p as

$$(\delta_0^2/\delta^2)\{F_{p-1}-(4p+1)F_p+2(p+1)(2p+1)F_{p+1}\}-(U\delta)'\{F_{p-1}+F_p-2(p+1)(2p+1)F_{p+1}\}/\delta-(UF_p)'=0 \tag{5.3}$$

Equation (5.3) forms an infinite set of linear coupled equations, the coupling however being restricted to the neighbouring equations only, unlike in the nonlinear problem where each equation contains all the F_p and their derivatives. By a proper choice of δ for the linear problem it is possible to decouple the equations further. If we choose δ such that

$$\delta(U\delta)'=\delta_0^2 \tag{5.4}$$

we find that the terms containing F_{p-1} in (5.3) are eliminated and we are left with F_p and F_{p+1} only. The equations will now be

$$(U\delta)'\{(2p+1)F_p-2(p+1)(2p+1)F_{p+1}\}/\delta+(UF_p)'=0\,. \tag{5.5}$$

It can be easily verified from (5.5) that the effect of any F_p is felt on only the lower components F_n, $n<p$. That is, the presence of F_2 at any x, for example, affects only the development of F_0 and F_1 but has no influence on the development of F_3, F_4 etc.

If the series (3.1) is truncated at $p=m$, the general solution for the set of equations (5.1) can be written as

$$UF_p(x)=\left\{\sum_{n=0}^{m-p}A_{p+n}\prod_{l=0}^{n}\lambda_{p+l-1}/(-2)^n n!\,(U\delta)^{2p+2n+1}\right\}$$
$$+\prod_{l=0}^{m-p}\lambda_{p+1}\int\frac{(U\delta)'}{(U\delta)^3}\int\frac{(U\delta)'}{(U\delta)^3}\cdots\int(U\delta)^{2m}UF_{m+1}(U\delta)'\,dx^{m-p+1}\,,$$
$$\lambda_n=2(n+1)(2n+1) \tag{5.6}$$

(we are *not* using the summation convention in this equation).

The A_n are constants to be evaluated from initial conditions. As noted earlier, if $F_p(0)=0$ for $p>m$ at the start it will remain so for all x, because the integral in (5.6) will be identically zero. For example if at $x=0$

$$F_3=F_4=\ldots=0$$

we get

$$UF_0=A_0/U\delta-A_1/(U\delta)^3+3A_2/(U\delta)^5$$
$$UF_1=A_1/(U\delta)^3-6A_2/(U\delta)^5$$
$$UF_2=A_2/(U\delta)^5 \tag{5.7}$$

Knowing the values of U, δ, F_0, F_1 and F_2 at the start A_0, A_1 and A_2 can be evaluated.

It can be seen from (5.6) that any component F_p decays like $(U\delta)^{-2p+1}$. Consequently, all profiles, irrespective of their initial shape, asymptotically tend towards the similarity profile D_0, which is the mode with the slowest decay rate of all.

The rate of growth of δ can now be found from (5.4) as

$$(\delta/\delta_0)=(U_0/U)\left\{1+(2/U_0^2)\int_0^x U(x)dx\right\}^{\frac{1}{2}}. \tag{5.8}$$

(5.8) is of course the same as the rate of wake spreading in the similarity solution for wakes [10].

5.1 *Particular Cases*

(a) Similarity profile: If we assume that the profile is exactly Gaussian ($F_n=0$, for all $n\neq 0$) we get the wellknown similarity solution [10]

$$F_0/F_0(0) = (U/U_0)\left\{1+(2/U_0^2)\int_0^x U(x)dx\right\}^{-\frac{1}{2}}. \tag{5.9}$$

(b) F_1 profile: In order to get a quantitative picture of the decay of a profile different from the similarity profile, a profile with $F_n(0) \neq 0$ only for $n=1$ is studied. For this case we get $F_2 = F_3 = \ldots = 0$ for all x. F_0, F_1 and the velocity profile in zero pressure gradient are given by

$$F_0 = F_1(0)(\delta_0/\delta - \delta_0^3/\delta^3)$$

$$F_1 = F_1(0)\delta_0^3/\delta^3$$

$$\frac{u(x,\eta)}{U_1} = 1 - \frac{\delta_0}{\delta}F_1(0)D_0(\eta) - \frac{\delta_0^3}{\delta^3}F_1(0)\left[D_0(\eta) - D_2(\eta)\right]. \tag{5.10}$$

Fig. 4 shows the evolution of the profile with x. It may be mentioned that Gold [11] has worked out the solutions for the development of an arbitrary profile (including the effect of compressibility) in the form of an integral. Even though Gold's solution is elegant in form, its numerical evaluation is rather involved. The present solution is much simpler, particularly when the number of components is not large as in the case of a perturbed similarity profile. Further any small change in the shape of a profile can be easily taken care of.

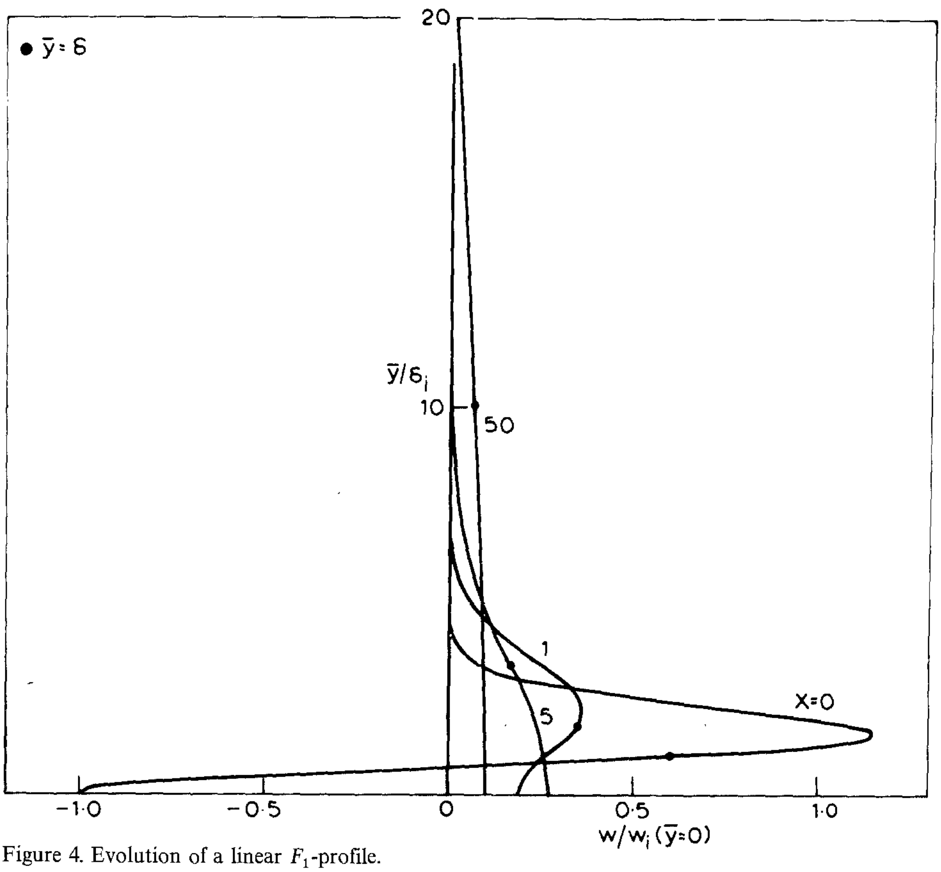

Figure 4. Evolution of a linear F_1-profile.

In the case of axisymmetric flow, we find that the decoupling cannot be effected to the same extent as in two-dimensional flow, due to the terms containing inverse powers of η on the right hand side of (2.6). The equation for any F_p contains all the $F_n (n > p)$ unlike the two-dimensional case where only F_{p+1} occurs in the equation for F_p. As a result the general solution for any F_p cannot be written out in simple form. However if $F_p = 0$ for $p > n$ the equations can be solved for all the F_p working backwards from the last equation as in the two-dimensional problem. If the number of terms is not large the solution will still be simple.

6. Concluding Remarks

A nonlinear mode theory is proposed and found to give solutions to great accuracy*. A series in parabolic cylinder functions is found to be an effective choice for the study of evolution of arbitrary wake profiles. When the maximum defect is not too close to unity, the first term approximation in the mode theory and the momentum integral solution both fetch results which are simple and satisfactory in many practical applications. The method, it appears, can be easily extended to other shear flows and also to turbulent flows if we assume a suitable eddy viscosity. These are being presently attempted.

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Appendix 1

Using the wellknown recurrence relations between parabolic cylinder functions [7], it can be shown from (3.4a) that (when $k=0$)

$$A^p_{mn} = (\tilde{A}^p_{mn} - 2n\tilde{A}^p_{m(n-1)})/2 \tag{A.1}$$

where $\tilde{A}^p_{mn}$ is defined as

$$\tilde{A}^p_{mn} = \int_0^\infty D_{2p} D_{2n+1} \left(\int_0^\eta D_{2m} d\eta \right) d\eta \bigg/ \int_0^\infty D^2_{2p} d\eta \,. \tag{A.2}$$

A few simple algebraic manipulations of (3.4) using the recurrence relations between parabolic cylinder functions yield the following relations between $\tilde{A}^p_{mn}$ and B^p_{mn}

$$\tilde{A}^p_{(m+1)n} = \tilde{A}^p_{mn} + 2m(2m-1)\tilde{A}^p_{(m-1)n} - 2B^p_{m(n+1)} - (2n+1)2B^p_{mn} \tag{A.3}$$

$$\begin{aligned}\tilde{A}^p_{mn} = {} & \tilde{A}^{(p+1)}_{m(n-1)}(2p+1)(2p+2) + (4p-4n+2)\tilde{A}^p_{m(n-1)} \\ & -2(n-1)(2n-1)\tilde{A}^p_{m(n-2)} + \tilde{A}^{(p-1)}_{m(n-1)}\end{aligned} \tag{A.4}$$

$$\tilde{A}^p_{mn} = \{\tilde{A}^n_{mp} + 2p\,\tilde{A}^n_{m(p-1)}\} \frac{(2n)!}{(2p)!} - 2n\,\tilde{A}^p_{m(n-1)} \tag{A.5}$$

$$B^p_{(m+1)n} = B^p_{m(n+1)} + 4(n-m)B^p_{mn} + 2n(2n-1)B^p_{m(n-1)} - 2m(2m-1)B^p_{(m-1)n} \tag{A.6}$$

$$B^p_{(m+1)n} = (2p+1)(2p+2)B^{p+1}_{mn} + 4(p-m)B^p_{mn} + B^{p-1}_{mn} - 2m(2m-1)B^p_{(m-1)n} \tag{A.7}$$

$$B^p_{mn} = B^p_{nm} \tag{A.8}$$

$$\tilde{A}^0_{mn} = \tilde{A}^0_{nm} = B^0_{mn} = B^0_{nm} \tag{A.9}$$

We find by exact integration that

$$\tilde{A}^0_{00} = (\tfrac{2}{3})^{\frac{1}{2}} \tag{A.10}$$

Knowing the value of $\tilde{A}^0_{\infty}$, the values of $\tilde{A}^p_{mn}$ and B^p_{mn} for any given value of m, n and p can be found by using equations (A.3) to (A.10) successively. $\tilde{A}^p_{mn}$ can then be found from (A.1). More details will be found in [14].

* We are informed by a referee that an apparently somewhat similar method had been suggested in an unpublished presentation by S.C.R. Dennis at an Agard meeting.

Appendix 2. Momentum Integral Solutions

It is found that, by using the momentum integral approach it is possible to obtain a simple, fairly accurate, closed form solution for the development of large defect wakes which have an approximate exponential form, as in the case of the wake behind a flat plate or behind symmetric bodies downstream of the recirculation region. It must, however, be realised that the solution is not amenable to improvement as in the exact method outlined in § 3.

Hill [12] has used this approach (together with an auxiliary moment of momentum equation) for studying wake problems. However, the problem has been analysed only for small defects in which case this approach is unnecessary as we have the exact solution even for arbitrary initial profiles. Later Hill [13] has used the momentum integral technique in turbulent flow of a jet in the presence of an outside stream or a duct, assuming that the velocity and shear stress profiles are the same self-similar distributions that have been determined by experiments on a *free* jet. However, we are not aware of a momentum integral treatment for the nonlinear laminar wake problem, and so give below a brief description.

The momentum integral equation for a wake in zero pressure gradient can be written as

$$\int_0^\infty u(1-u)y^k\,dy = \text{constant}\,. \tag{B.1}$$

Substituting for u from (3.1) and retaining only F_0 (assuming a Gaussian profile for all x) we get

$$\delta^{k+1}\int_0^\infty (1-F_0D_0)F_0D_0\eta^k\,d\eta = \text{constant} \tag{B.2}$$

which on simplification yields

$$F_0(1-\alpha_0F_0) = \Lambda_1/\delta\,, \qquad k=0 \tag{B.3a}$$

$$F_0(1-\beta_0F_0) = \Lambda_2/\delta^2\,, \qquad k=1 \tag{B.3b}$$

where

$$\alpha_0 = \int_0^\infty D_0^2\,d\eta \Big/ \int_0^\infty D_0\,d\eta \tag{B.4a}$$

$$\beta_0 = \int_0^\infty D_0^2\eta\,d\eta \Big/ \int_0^\infty D_0\eta\,d\eta\,, \tag{B.4b}$$

Λ_1 and Λ_2 are constants evaluated at $x=0$ using (B.3a, b).

(B.3) provides one equation relating F_0 and δ. The other equation between F_0 and δ is taken from the centre line condition (3.9). The set of equations (4.2) and (B.3) yield

$$(\nu\bar{x}/U_1\Lambda_1^2) = (1-\alpha_0F_0)^2/2F_0^2 + (3\alpha_0-1)(1-\alpha_0F_0)/F_0$$
$$+ (3\alpha_0^2-2\alpha_0)\ln\{(1-\alpha_0F_0)/F_0\} + \{(\alpha_0^2-\alpha_0^3)F_0/(1-\alpha_0F_0)\} + C_1,\ k=0\,;$$
$$(\nu\bar{x}/U_1\Lambda_2) = (1-\beta_0F_0)/2F_0 + \{(\beta_0-1)/2\}\ln\{(1-\beta_0F_0)/F_0\} + C_2, \quad k=1\,. \tag{B.5a, b}$$

where C_1, C_2 are arbitrary constants of integration.

Figure 2 shows a comparison of (B.5a) with the more exact solutions in two-dimensional flow. Figure 3 compares the solution (B.5b) with the one-term solution obtained in § 4 for axisymmetric wake flow; the agreement is seen to be very close.

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